

1. Facility Location

1.1 The Supply Chain

Figure 1.1 shows a typical supply chain for a single plant. In the figure, items D and E provided by two different tier-two suppliers are shipped to a tier-one supplier where they are used to produce item B, units of which are then shipped to the plant. Similarly, items F and G are used to produce item C, which, along with B, are the two raw materials used to produce the finished good A. Units of A are shipped to distribution centers (DCs), from which they are delivered to the final customer. The number of units shown along each *lane* (or arc) in the figure is meant to indicate the typical size of each shipment; e.g., one of the reasons that a DC might be used is that it is cheaper to transport larger loads from the plant to the DC and then transport small loads to each customer as needed.

1.1.1 Logistics Network Design

The design of a logistics network involves determining how to supply products to customers at the least cost while providing the desired level of service. The level of service can include both the time required for delivery to the customer and the availability of the product. Given the location of available transshipment points (e.g., ports and DCs), the design problem includes issues like selecting the best mode of transportation between each point in the network, the frequency and quantity of each shipment, and the amount of each product to be stored at each DC. The total cost of transportation and inventory is used to guide the design, subject to service level requirements. Inventory costs are the sum of cycle, in-transit, and safety stock inventory. Generally, there is a trade-off between transportation and inventory costs; e.g., transportation costs are lower for truckload shipments as compared to less-than-truckload, but they can increase cycle inventory costs while product waits for a full truckload to accumulate. In some cases, the design problem includes the need to select the best location for some of the DCs in the network. When location decisions are part of the design, it may be necessary to include the cost of constructing and operating each DC as part of the total cost along with the cost of transportation and inventory.

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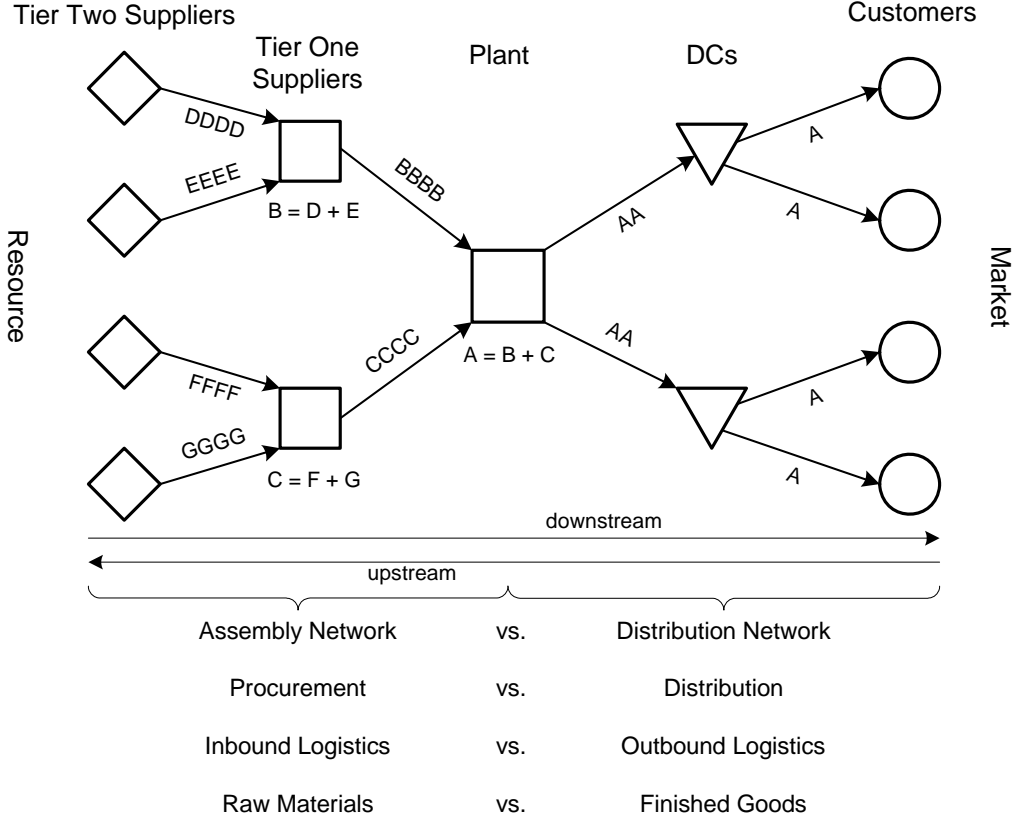


Figure 1.1. Typical logistics network for a plant.

As an example of a network design problem in the furniture industry, assume that a national furniture manufacturer has a large import program from suppliers in China. Currently, all products are shipped through the Panama Canal to a port on the East Coast and then trucked to a single DC. From there product is shipped to retail customers throughout the continental U.S. The manufacturer is considering whether a second DC should be opened at a predetermined location in California and used to receive product through the port of Long Beach. The second DC will decrease transportation costs from the DCs to the customers, but the sum of the inventory held both DCs will need to be greater because of loss of the opportunity at the single DC to pool safety stock inventory. Also, transport costs from China to Long Beach will be lower than the costs to the East Coast, but it may require than full container loads of low-volume products be shipped to each DC as compared to single container loads to the single DC, thereby doubling the cycle inventory levels of these products. Given all these factors, a network design decision can be made by determining the change in total logistics costs (transportation plus inventory costs) associated with opening the second DC.

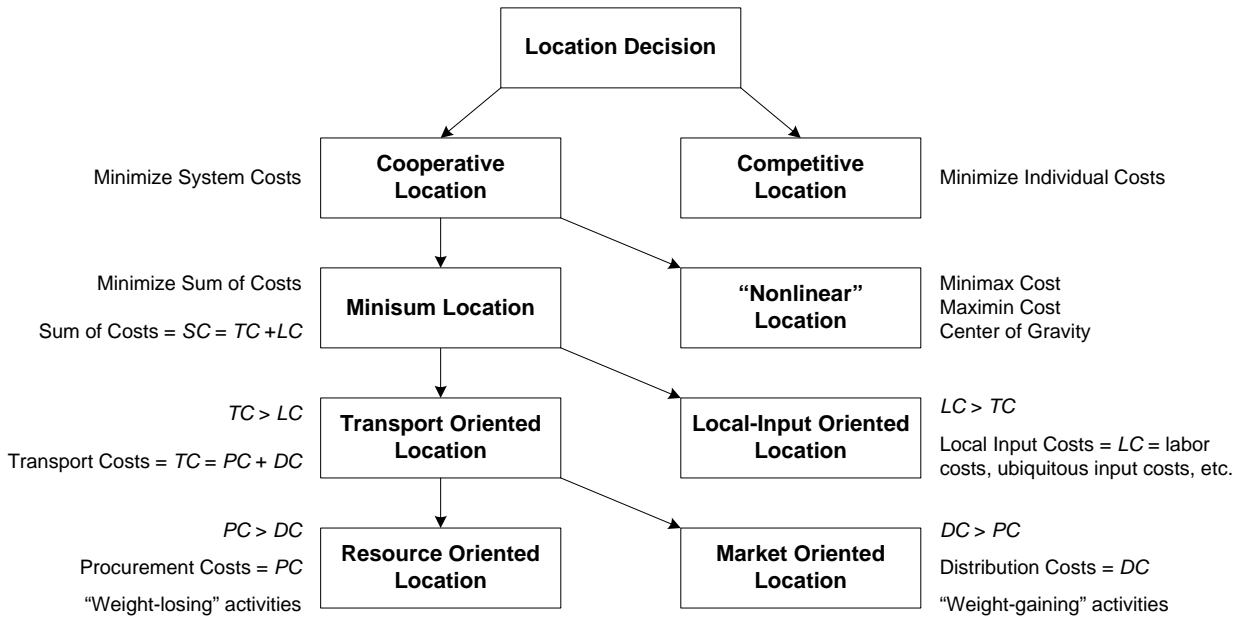


Figure 1.2. Types of location problems.

1.1.2 Location Problems

A taxonomy of the different types of location problems is shown in Figure 1.2. Cooperative location decisions are used to minimize the total system costs of multiple facilities owned by a single firm instead just minimizing the cost of each of the firm's individual facilities. These decisions are possible when the impact of the location of other firms' facilities does not significantly impact the location of your facilities. Cooperative location problems that focus on optimizing something other than the sum of costs (e.g., minimizing the maximum cost) can be considered as "nonlinear" location problems because the location-related costs in these problems are not directly proportional to distance, as is the case in minisum problems. Competitive location decisions are used to minimize the cost of an individual facility with respect to other facilities owned by other firms. These decisions are required when the location of other firms' facilities does impact the location of your facilities, and may result in sub-optimal decisions as compared to cooperative location decisions (cf. Hotelling's law). In what follows, only transport-oriented minisum location problems are considered because these problems are the ones that most benefit from a simple analysis using transport-cost minimization as the sole criterion, and the assumption that costs are directly proportional to distance is usually reasonable; local-input-oriented location problem are typically solved using more complex multi-criteria-based approaches.

1.1.3 Basic Production System

As shown in Figure 1.3, a production system can be considered as a node (or facility) in a logistics network that converts raw materials procured from suppliers into finished goods that are distributed to customers. For most production systems, the material input to the system equals

the material output from the system. *Raw materials* are those inputs that are transported to the production facility; *ubiquitous inputs* (e.g., water) are those available at any location, so that they do not need to be transported. *Finished products* are those outputs transported from the facility; while *scrap* is the output that is disposed of locally (although some outputs termed “scrap” are sometimes transported long distances from the facility for disposal or rework).

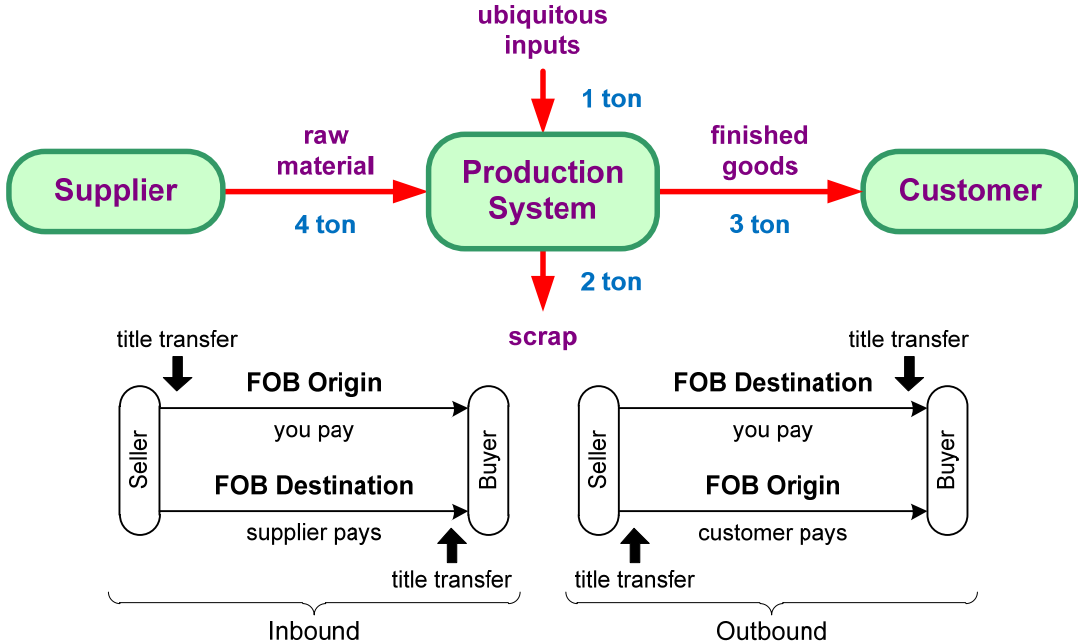


Figure 1.3. Basic production system.

The bottom portion of Figure 1.3 illustrates two different shipping terms that describe when the transfer of title occurs when goods are transported from the seller to the buyer: *FOB Origin* and *FOB Destination*, where FOB stands for free on board.¹ In most cases, the cost of transporting the goods is paid for by whoever is the owner of the goods during the transport. Referring to Figure 1.3, assuming that you represent the production system, the supplier (the seller) would pay for the transport of goods from the supplier’s location (the origin) to your (the buyer’s) facility (the destination) if the shipping terms were *FOB Destination*.

1.2 Single-Facility Minisum Location

Assuming that local input costs are either the same at every location or are insignificant as compared to transport costs, the minisum transport-oriented single-facility location problem is to locate a new facility (NF) to minimize the sum of weighted distances between NF and m existing facilities EF_i , $i = 1, \dots, m$:

$$\text{Min } TC(X) = \sum_{i=1}^m w_i d(X, P_i) = \sum_{i=1}^m \underbrace{q_i r_i}_{w_i} \overbrace{d(X, P_i)}^{c_i} \quad (1.1)$$

where

w_i = monetary weight (\$/mile)

q_i = physical weight (tons) or f_i = physical weight rate (tons/year)

r_i = transport rate (\$/ton-mile)

$d(X, P_i)$ = distance between NF at X and EF_i at P_i (miles)

c_i = unit cost (\$/ton), used to determine q_i after NF located (transportation problem)

X = location of new facility (NF)

P_i = location of existing facility i (EF_i)

m = number of EFs

If physical weight q_i is used, then TC in (1.1) is in units of \$; if, instead, the physical weight rate f_i in units of tons per year is used, then TC is in units of \$/year.

1.2.1 Majority Theorem

The Majority Theorem can be used to determine if one of the EFs has at least half of the total weight (i.e., a majority); if so, then the NF should be located at that EF in order to minimize TC :

$$\text{Majority Theorem:} \quad \text{Locate NF at } EF_j \text{ if } w_j \geq \frac{W}{2}, \quad \text{where } W = \sum_{i=1}^m w_i \quad (1.2)$$

The theorem is true for all minisum problems with metric distances, and can be used as a first check before using other means of determining the optimal location for the NF.

1.2.2 Weight-Gaining vs. Weight-Losing Activities

The activity occurring at a NF is considered *weight losing* if the sum of the monetary weights from EFs supplying material to the NF exceeds the sum of the weights of the material sent from the NF to the EFs which it supplies; conversely, if the sum of the monetary weight into the NF is less than the sum of the weight out, then the activity is considered *weight gaining*. In situations

where the NF is a distribution center (DC), it is common for the products to be weight gaining because, while the physical weight of the products into the DC is the same as the weight of the products out, more costly modes of transport are used for distribution as compared to procurement, resulting in higher monetary weights on the outbound side of the DC (and thus drawing the location of the DC to the market).

1.2.3 Median Location

Starting from the first EF, a median location is the first EF location at which the cumulative weight of the EFs up to that point is at least half of the total weight of all EFs (i.e., the EF location that splits the total weight into equal halves). The median location is the optimal NF location for all 1-D minisum problems and any 2-D rectilinear distance location problem.

The following procedure can be used to determine the median location:

Median location: For each dimension x of X :

1. Order EFs so that $|x_1| \leq |x_2| \leq \dots \leq |x_m|$

2. Locate x -dimension of NF at the first EF j where $\sum_{i=1}^j w_i \geq \frac{W}{2}$, where $W = \sum_{i=1}^m w_i$

If the cumulative weight at EF j exactly equals half of the total weight, then the optimal location along the x -dimension for the NF is any point between and including EF j and EF $(j+1)$. Note that the optimal location for the NF can be determined without knowing the actual distances between the EFs; all that is necessary is to be able to order the EFs along each dimension.

For 2-D minisum problems other than rectilinear distance problems, then an iterative procedure must be used to optimize TC and thus determine the optimal location for a NF.

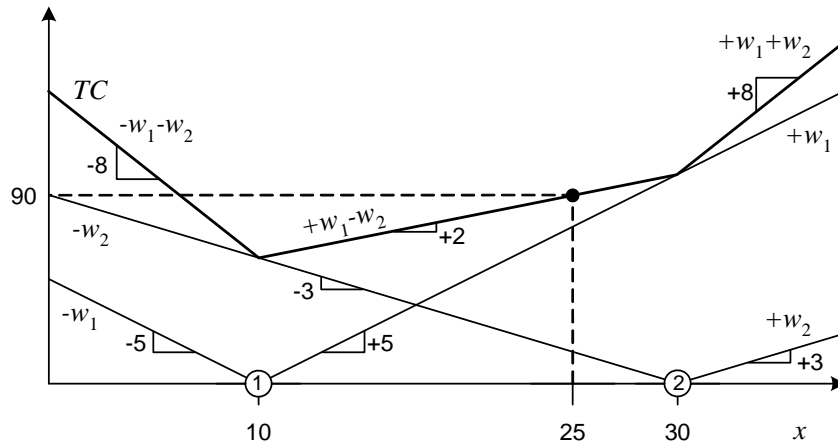


Figure 1.4. Total cost curve for 2 EFs.

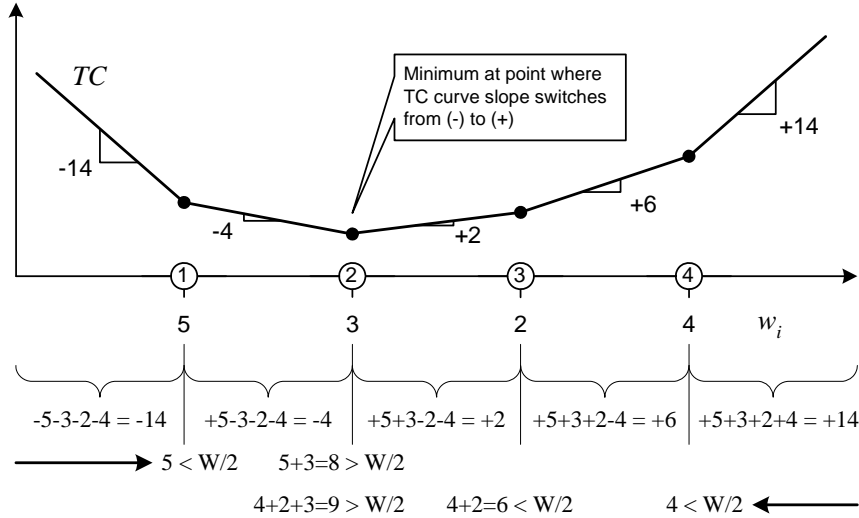


Figure 1.5. Total cost curve for 4 EFs.

Derivation

Figure 1.4 shows an example with two EFs. EF1 is located at $x_1 = 10$ and has a weight of $w_1 = 5$, while EF2 is located at $x_2 = 30$ and has a weight of $w_2 = 3$. The total cost of locating a single NF at x is the sum of the costs for each EF:

$$TC(x) = \beta_1(x - x_1) + \beta_2(x - x_2), \quad \text{where } \beta_i = \begin{cases} w_i, & \text{if } x \geq x_i \\ -w_i, & \text{if } x < x_i \end{cases}$$

For NF at $x = 25$,

$$\begin{aligned} TC(25) &= w_1(25 - 10) + (-w_2)(25 - 30) \\ &= 5(15) + (-3)(-5) = 90 \end{aligned}$$

Figure 1.5 shows an example with four EFs, where only the total cost curve is shown. The slope of curve from $-\infty$ to x_1 is

$$\beta_{-\infty,1} = -\sum_{i=1}^m w_i = -W$$

and the slope between x_j and x_{j+1} is

$$\beta_{j,j+1} = \sum_{i=1}^j w_i - \sum_{i=j+1}^m w_i$$

Starting at x_1 , the minimum total cost corresponds to the point where the slope of the total cost curve switches from negative to positive (or zero); this is equivalent to finding the first j such that

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$$\begin{aligned}\sum_{i=1}^j w_i - \sum_{i=j+1}^m w_i &\geq 0 \\ \sum_{i=1}^j w_i &\geq \sum_{i=j+1}^m w_i \\ \sum_{i=1}^j w_i + \sum_{i=1}^j w_i &\geq \sum_{i=1}^j w_i + \sum_{i=j+1}^m w_i \\ 2\sum_{i=1}^j w_i &\geq W \\ \sum_{i=1}^j w_i &\geq \frac{W}{2}\end{aligned}$$

In Figure 1.5, $W = 14$ and $W/2 = 7$. Starting from either EF1 or EF4, the optimal solution is found at EF2, which corresponds to the location at which the median condition is first satisfied.

1-D Example

Problem: As shown in Figure 1.6, I-40 passes through Asheville, Statesville, Winston-Salem, Greensboro, Durham, Raleigh, and Wilmington. The number of road miles from the beginning of I-40 at the western border of North Carolina to each city is shown below its name. A company wants to build a facility along I-40 to serve customers located in these cities. If the weekly demand in truckloads of the customers in each city is 6, 4, 3, 2, 1, 3, and 5, respectively, determine where the facility should be located to minimize the distance traveled to serve the customers assuming that I-40 will be used for all travel.

Solution: Since $W = 24$, the cumulative weights at the optimal location should equal or exceed $W/2 = 12$, which occurs at Winston-Salem. Note that (1) the same solution is found starting from either Asheville or Wilmington, and (2) the distance between cities is not used to determine the optimal location, just the relative ordering the cities along I-40.

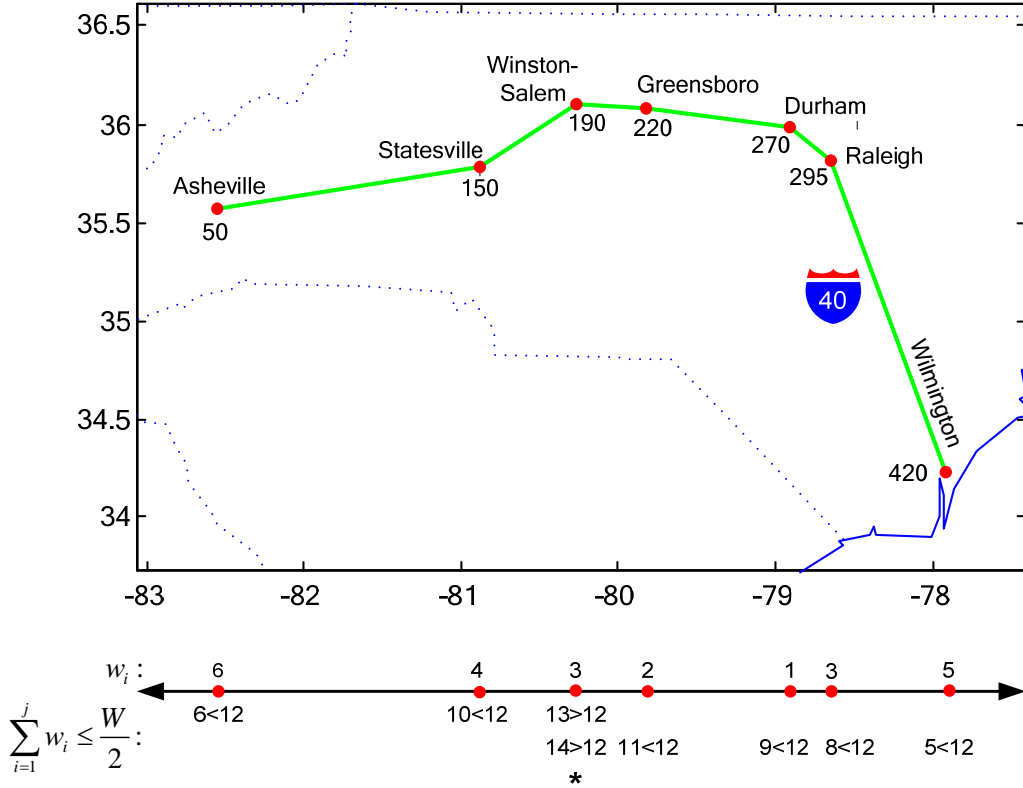


Figure 1.6. 1-D minisum location example.

2-D Example

Problem: A new snack machine is to be located on the floor of a facility (see Figure 1.7). Workers from eight different departments will make 19, 53, 82, 42, 9, 8, 39, and 6 trips per shift to the machine. Assuming rectilinear distance is a reasonable approximation of the actual travel distance, what is the location for the snack machine that will minimize the total distance that the workers have to travel?

Solution: A separate 1-D location problem can be solved for each dimension. Since $W = 258$, the cumulative weights at the optimal location should equal or exceed $W/2 = 129$. Along the bottom (x) dimension, the optimal location is at same x -location as departments 4 and 6, while along the left (y) dimension, the optimal location is anywhere between y -locations of departments 4 and 6. The optimal y -location is not a single point because the cumulative weight exactly equals 129, which corresponds to the total cost curve being flat between points 4 and 6. Note that the weights of departments at the same location are added together along each dimension.

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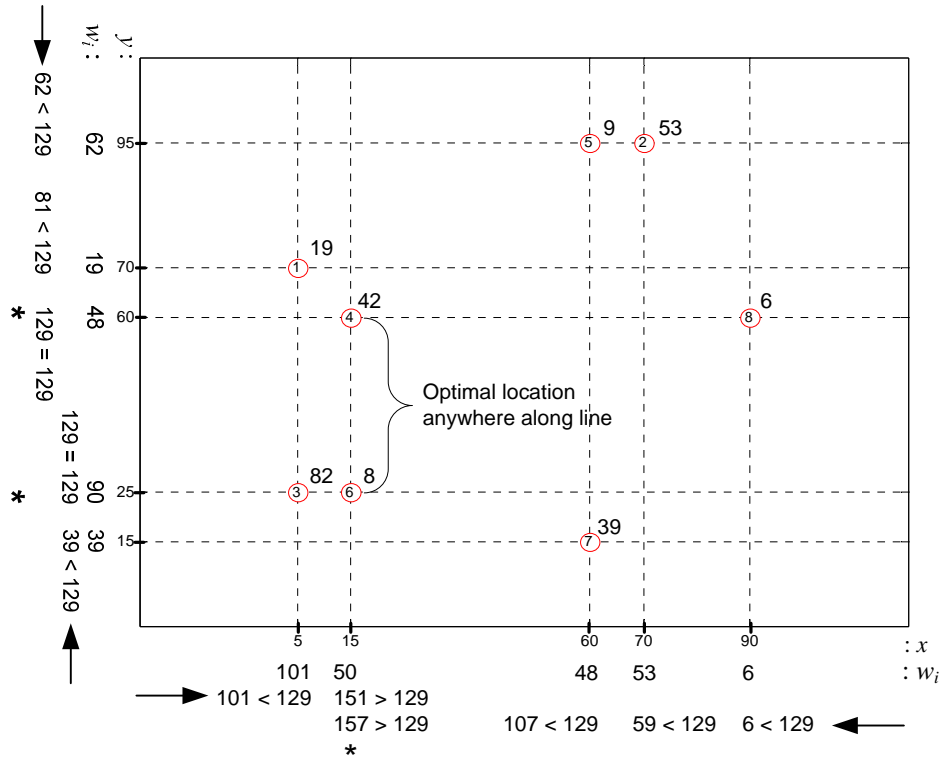


Figure 1.7. 2-D rectilinear-distance minisum location example.

1-D Example with Procurement and Distribution Costs

Problem: A product will be produced at a single plant that will be located along I-40 to serve customers located in the cities shown in Figure 1.6, above. Two tons of raw materials from a supplier in Asheville and a half ton of a raw material from a supplier in Durham are used to produce each ton of finished product that is shipped to customers in Statesville, Winston-Salem, and Wilmington. The demand of these customers is 10, 20, and 30 tons per year, respectively, and it costs \$0.33 per ton-mile to ship raw materials to the plant and \$1.00 per ton-mile to ship finished goods from the plant to the customers. Determine where the plant should be located so that procurement and distribution costs (i.e., the transportation costs to and from the plant) are minimized.

Solution: As shown in Figure 1.8, total customer demand is 60 tons per year, which translates into a demand of 120 tons per year based on a bill-of material (BOM) ratio of 2 for the supplier in Asheville and 30 tons per year based on a BOM of 0.5 for the supplier in Durham. The resulting monetary weights are used to determine that the plant should be located in Winston-Salem. The plant is (monetary) weight gaining since $\Sigma w_{in} = 50 < \Sigma w_{out} = 60$, but the plant is physically weight losing since $\Sigma f_{in} = 150 > \Sigma f_{out} = 60$; the difference is due the higher outbound transport rate.

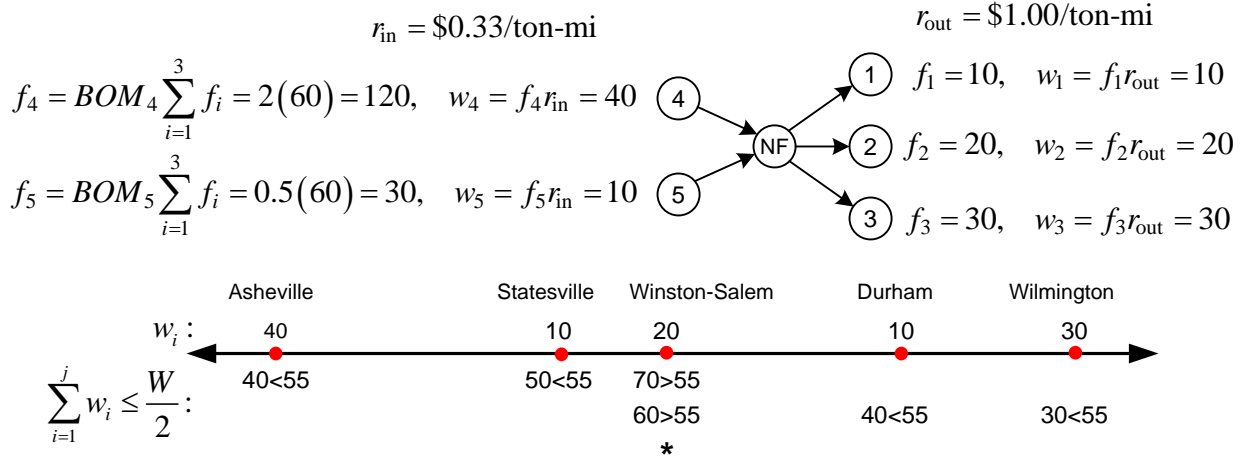


Figure 1.8. 1-D example with procurement and distribution costs.

1.3 Metric Distances

1.3.1 l_p Distances

Given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, a general metric distance function termed the l_p distance or norm can be used to represent several of the most common distances used in practice. The function uses different values of the parameter p to represent these special cases:

$$\text{General } l_p: \quad d_p(P_1, P_2) = \left[|x_1 - x_2|^p + |y_1 - y_2|^p \right]^{\frac{1}{p}}, \quad p \geq 1 \quad (1.3)$$

$$\text{Rectilinear: } \underset{(p=1)}{d_1(P_1, P_2)} = |x_1 - x_2| + |y_1 - y_2| \quad (1.4)$$

$$\text{Euclidean: } \underset{(p=2)}{d_2(P_1, P_2)} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (1.5)$$

$$\text{Chebychev: } \underset{(p \rightarrow \infty)}{d_\infty(P_1, P_2)} = \max \{ |x_1 - x_2|, |y_1 - y_2| \} \quad (1.6)$$

The above distances are defined for two-dimensional points, but they can be extended for points of any dimension. For $0 < p < 1$, the l_p distance is not defined since the triangle inequality does not hold; for $1 < p < 2$, the l_p distance lies between the rectilinear and the Euclidean distances; and for $2 < p < \infty$, the l_p distance lies below the Euclidean distance and decreases as p increases.

Proof of Chebychev Distance

Without loss of generality, let $P_1 = (x, y)$, for $x, y \geq 0$, and $P_2 = (0, 0)$. Then $d_\infty(P_1, P_2) = \max \{x, y\}$ and $d_p(P_1, P_2) = \left[x^p + y^p \right]^{1/p}$.

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If $x = y$, then $\lim_{p \rightarrow \infty} [x^p + y^p]^{1/p} = \lim_{p \rightarrow \infty} [2x^p]^{1/p} = \lim_{p \rightarrow \infty} [2^{1/p} x] = x$.

If $x < y$, then $\lim_{p \rightarrow \infty} [x^p + y^p]^{1/p} = \lim_{p \rightarrow \infty} \left[\left((x/y)^p + 1 \right) y^p \right]^{1/p} = \lim_{p \rightarrow \infty} \left((x/y)^p + 1 \right)^{1/p} y = 1 \cdot y = y$.

A similar argument can be made if $x > y$. ■

1.3.2 Great Circle (Geodesic) Distances

Great circle, or geodesic, distances on the surface of a sphere (e.g., the earth (see Figure 1.9)) correspond to the shortest distance between two points on the surface along the circle formed by the intersection of the surface and a plane passing through the center of the sphere (i.e., a “great circle”). The elevations of the points on the surface are usually ignored.

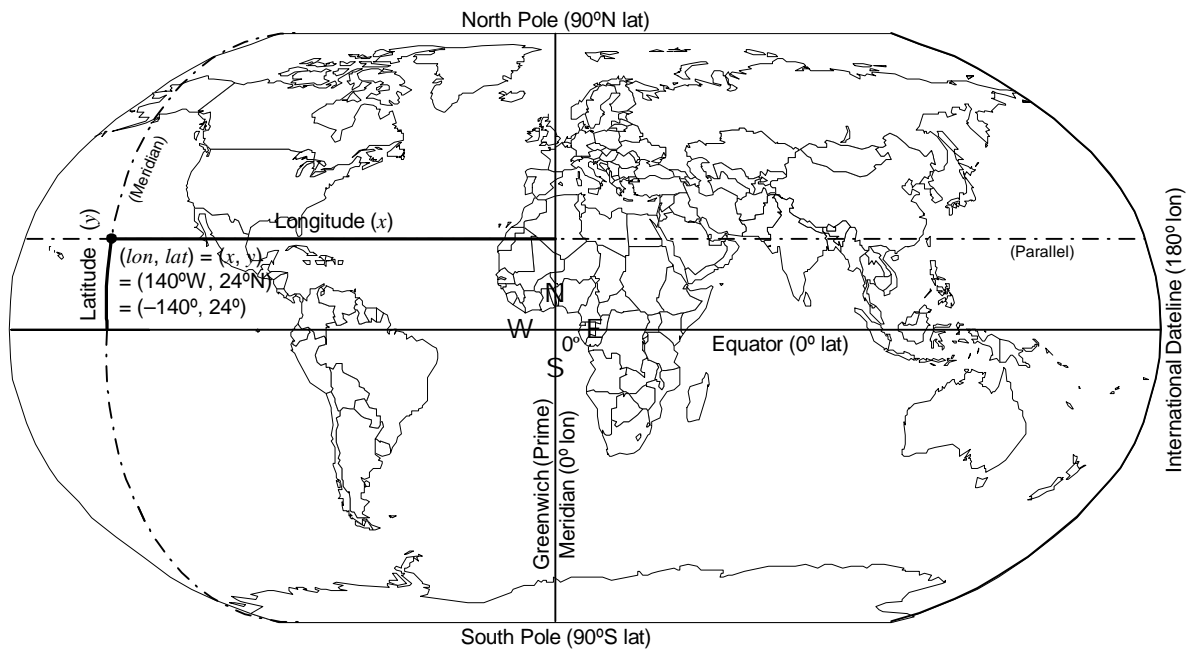


Figure 1.9. Longitude and latitude for points on the surface of the earth.

The great circle distance (d_{GC}) between points 1 and 2 on the surface of the earth, specified by their longitude (lon) and latitude (lat) angles (in radians), is as follows:

$$(lon_1, lat_1) = (x_1, y_1), \quad (lon_2, lat_2) = (x_2, y_2)$$

$$d_{rad} = (\text{great circle distance in radians of a sphere})$$

$$= \cos^{-1} \left[\sin y_2 \sin y_1 + \cos y_2 \cos y_1 \cos(x_1 - x_2) \right]$$

$$R = (\text{radius of earth at equator}) - (\text{bulge from north pole to equator})$$

$$= 3,963.34 - 13.35 \sin\left(\frac{y_1 + y_2}{2}\right) \text{ mi}, \quad = 6,378.388 - 21.476 \sin\left(\frac{y_1 + y_2}{2}\right) \text{ km}$$

$$d_{GC} = \text{distance } (x_1, y_1) \text{ to } (x_2, y_2) = \boxed{d_{rad} \cdot R}$$

To convert between decimal degrees and radians: $x_{rad} = \frac{x_{deg}}{180} \pi$ and $x_{deg} = \frac{x_{rad} \cdot 180}{\pi}$

To convert degrees ($DD:MM:SS$) to decimal degrees: $x_{deg} = \begin{cases} DD + \frac{MM}{60} + \frac{SS}{3,600}, & \text{if } E \text{ or } N \\ -DD - \frac{MM}{60} - \frac{SS}{3,600}, & \text{if } W \text{ or } S \end{cases}$

Example

Table 1.1 shows a spreadsheet that calculates the great circle distance from Raleigh, NC to Gainesville, FL, Baghdad, Iraq, and Rio de Janeiro, Brazil. There are typically slight differences in great circle distance calculations due to how accurately the radius of the earth, R , is determined.

Table 1.1. Great Circle Distances from Raleigh

	dd	mm	ss		x (deg)	x (rad)	dd	mm	ss		y (deg)	y (rad)	d(rad)	d(mi)
Raleigh	78	39	32	W	-78.659	-1.373	35	49	19	N	35.8219	0.6252		
Gainesville	82	20	11	W	-82.336	-1.437	29	40	27	N	29.6742	0.5179	0.12	475.0745
Baghdad	44	22		E	44.3667	0.7743	33	14		N	33.2333	0.58	1.62	6407.166
Rio de Janeiro	43	12		W	-43.2	-0.754	22	57		S	-22.95	-0.401	1.181	4679.089

Circuitry Factor

Since the great circle distance is usually less than the actual road (or rail) distance between any two locations, the great circle distance can be multiplied by a circuitry factor k to approximate the actual road distance:

$$\text{Actual road distance between } P_1 \text{ and } P_2 \approx k \cdot d_{GC}(P_1, P_2)$$

The circuitry factor is the average ratio of the actual road distance between two points and the great circle distance between the two points. It can be used to increase the easy-to-determine great circle distance so that it approximates the harder-to-determine actual road distance. You can use the websites *How far is it* or *Google Maps* to determine the great circle distances and the latter to determine road distances.

Circuitry factors of 1.15 to 1.25 are typically used for long-distance (> 20 mile) approximations, while factors of 1.25 to 1.50 are typically used for short-distance road approximations and for rail networks because the networks are not as dense. A factor of 1.20 provides a reasonable long-distance road approximation for the continental U.S.

Derivation

In Figure 1.10, a , b , and c are the sides of a spherical triangle and A , B , and C are the corresponding angles. The great circle distance between points 1 and 2 in radians (d_{rad}) corresponds to side c of the triangle.

Given

$$a = 90^\circ - y_2, \quad b = 90^\circ - y_1, \quad C = x_1 - x_2$$

the formula for great circle distances can be derived using the *spherical law of cosines for sides*:

$$\begin{aligned} \cos(c) &= \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C) \\ &= \cos(90^\circ - y_2)\cos(90^\circ - y_1) + \sin(90^\circ - y_2)\sin(90^\circ - y_1)\cos(x_1 - x_2) \\ &= \sin(y_2)\sin(y_1) + \cos(y_2)\cos(y_1)\cos(x_1 - x_2) \end{aligned}$$

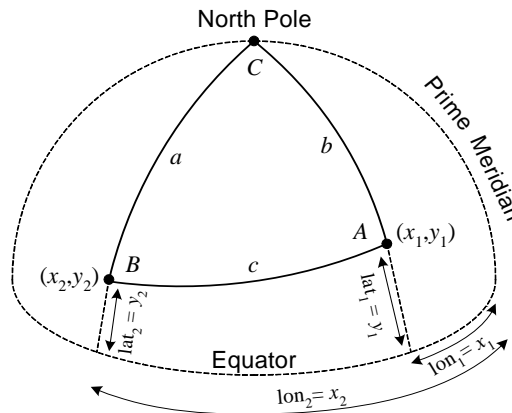


Figure 1.10. Great circle distance derivation.

Solving for c :

$$d_{rad} = c = \cos^{-1} \left[\sin(y_2)\sin(y_1) + \cos(y_2)\cos(y_1)\cos(x_1 - x_2) \right]$$

The above formula can result in round off error if the two points are located at exactly opposite sides of a sphere. Instead, the Haversine formula can be used:²¹

$$d_{rad} = c = 2 \sin^{-1} \min \left\{ 1, \sqrt{\sin^2 \left(\frac{y_1 - y_2}{2} \right) + \cos(y_1)\cos(y_2)\sin^2 \left(\frac{x_1 - x_2}{2} \right)} \right\}$$

The Haversine formula does not need to be used if the great circle distance is being calculated by hand and the two points are known to not be on opposite sides of the sphere.

1.4 Multifacility Location

In a multifacility location problem, the number of NFs to be located can either be specified or can be determined as part of the location procedure. When the number of NFs is specified, the allocation of EFs to NFs can either be given or determined as part of what is then termed a *location–allocation* problem. If the NF-to-EF allocations are given and there are no interactions between the NFs, then the multifacility problem reduces to a series of single-facility location problems. The location–allocation problem remains difficult even when there are no interactions between the NFs because of the need to determine the allocations.

Determining the best retail warehouse locations is an example of a location–allocation problem, where the EFs are population centroids (e.g., ZIP codes). Table 1.2 lists the best locations for a given number of warehouses. It is assumed that the warehouses serve retail customers located throughout the continental U.S. in proportion to population. Trucks traveling at 400 miles per day are used for all transport. Only outbound transport costs are used in making the location decision; it is reasonable to ignore inbound transport as long as suppliers are located uniformly throughout the country so that the inbound transport costs to each warehouse is approximately the same at any location. As can be seen in the table, the best single warehouse location (Bloomington, Indiana) is not the best location as the number of warehouses increases, while the warehouse in Palmdale, California remains in the best west coast location until a second west coast warehouse in Tacoma, Washington is added as part of the five-warehouse solution.

Table 1.2. Best Retail Warehouse Locations³

Number of Locations	Average Transit Time (days)	Warehouse Location		
1	2.20	Bloomington, IN		
2	1.48	Ashland, KY	Palmdale, CA	
3	1.29	Allentown, PA	Palmdale, CA	McKenzie, TN
4	1.20	Edison, NJ	Palmdale, CA	Chicago, IL
		Meridian, MS		
5	1.13	Madison, NJ	Palmdale, CA	Chicago, IL
		Dallas, TX	Macon, GA	
6	1.08	Madison, NJ	Pasadena, CA	Chicago, IL
		Dallas, TX	Macon, GA	Tacoma, WA
7	1.07	Madison, NJ	Pasadena, CA	Chicago, IL
		Dallas, TX	Gainesville, GA	Tacoma, WA
		Lakeland, FL		
8	1.05	Madison, NJ	Pasadena, CA	Chicago, IL
		Dallas, TX	Gainesville, GA	Tacoma, WA
		Lakeland, FL	Denver, CO	
9	1.04	Madison, NJ	Alhambra, CA	Chicago, IL
		Dallas, TX	Gainesville, GA	Tacoma, WA
		Lakeland, FL	Denver, CO	Oakland, CA
10	1.04	Newark, NJ	Alhambra, CA	Rockford, IL
		Palistine, TX	Gainesville, GA	Tacoma, WA
		Lakeland, FL	Denver, CO	Oakland, CA
		Mansfield, OH		

1.4.1 The Uncapacitated Facility Location Problem

Given m EFs and n sites at which NFs can be established, the uncapacitated facility location (UFL) problem can be formulated as the following mixed-integer linear programming (MILP) problem:

$$\text{Min } TC = \sum_{i=1}^n k_i y_i + \sum_{i=1}^n \sum_{j=1}^m c_{ij} x_{ij} \quad (1.7)$$

subject to

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, \dots, m \quad (1.8)$$

$$y_i \geq x_{ij}, \quad i = 1, \dots, n; j = 1, \dots, m \quad (1.9)$$

$$0 \leq x_{ij} \leq 1, \quad i = 1, \dots, n; j = 1, \dots, m \quad (1.10)$$

$$y_i \in \{0, 1\}, \quad i = 1, \dots, n, \quad (1.11)$$

where

k_i = fixed cost of establishing a NF at site i

c_{ij} = variable cost to serve all of EF j 's demand from site i

$y_i = 1$, if NF established at site i ; 0, otherwise

x_{ij} = fraction of EF j 's demand served from NF at site i .

The UFL problem is a MILP because the y_i 's are binary variables and the x_{ij} 's are real variables. In the UFL problem, all x_{ij} are 0 or 1; in the capacitated facility location (CFL) problem, there is a maximum capacity associated with each site, resulting in a x_{ij} value between 0 and 1 whenever not all of an EF j 's demand can be served from the NF at site j .

The MILP for the UFL problem is termed the “strong formulation” due to constraints (1.9), which results in an LP relaxation that gives a tight lower bound. When nm constraints (1.9) are replaced with the n constraints

$$m y_i \geq \sum_{j=1}^m x_{ij}, \quad i = 1, \dots, n, \quad (1.12)$$

the formulation is termed “weak” since the lower bound from the LP relaxation is not very tight, resulting in a large branch-and-bound tree.

Example

Given $n = 6$ sites, $m = 6$ EFs, and fixed and variable costs

$$[k_i] = \begin{bmatrix} 8 \\ 8 \\ 10 \\ 8 \\ 9 \\ 8 \end{bmatrix}; \quad [c_{ij}] = \begin{bmatrix} 0 & 3 & 7 & 10 & 6 & 4 \\ 3 & 0 & 4 & 7 & 6 & 7 \\ 7 & 4 & 0 & 3 & 6 & 8 \\ 10 & 7 & 3 & 0 & 7 & 8 \\ 6 & 6 & 6 & 7 & 0 & 2 \\ 4 & 7 & 8 & 8 & 2 & 0 \end{bmatrix},$$

the optimal solution is to establish two NFs at sites 3 and 6 serving EFs 2–4 and EFs 1, 5, and 6, respectively, for a $TC = 31$:

$$[y_i] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad [x_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

1.4.2 The p -Median Problem

When the number of NFs is specified and all of fixed costs are identical (or not stated), then the fixed costs will have no impact on the location decision and can be set to zero in (1.7) and the following constraint can be added to the UFL problem to formulate the p -median problem:

$$\sum_{i=1}^n y_i = p, \quad (1.13)$$

where p is the number of NFs to be located. If non-identical fixed are included, then the p -median problem generalizes to the p -UFL problem, where both p -median and UFL are special cases.⁴

1.5 References

The following sources are recommended for further study:

- Ahuja, R.K., Magnanti, T.L., and Orlin, J.B., 1993, *Network Flows: Theory, Algorithms, and Applications*, Englewood Cliffs, NJ: Prentice-Hall.
- Ballou, R.H., 2004, *Business Logistics/Supply Chain Management*, 5th Ed., Englewood Cliffs, NJ: Pearson.
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- Larson, R.C., and Odoni, A.R., 1981, *Urban Operations Research*, Englewood Cliffs, NJ: Prentice-Hall.
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- Simchi-Levi, D., Kaminsky, P., and Simchi-Levi, E., 2008, *Designing and Managing the Supply Chain: Concepts, Strategies, and Case Studies*, 3rd Ed., Boston: McGraw-Hill.

Notes

¹ *What are FOB shipping terms?* <http://simplestudies.com/what-are-fob-shipping-terms.html> (accessed Jan 2012).

² Sinnott, R.W., 1984, "Virtues of the Haversine", *Sky and Telescope* 68(2):159, as reported in <http://www.census.gov/cgi-bin/geo/gisfaq?Q5.1>

³ Table is adapted from *Inbound Logistics*, Oct. 2004, p. 50.

⁴ Krarup, J., and Pruzan, P.M., "Ingredients of locational analysis," in Mirchandani, P.B., and Francis, R.L., Eds., 1990, *Discrete Location Theory*, New York: Wiley, p. 20.

Facility Location–Allocation Problem

Location–Allocation (LA) Problem: Determine both the location of n new facilities (NFs) and the allocation of the flow requirements of m existing facilities (EFs) to the NFs that minimize total transportation costs.

Continuous: Minimize $f(X, W) = \sum_{i=1}^m \sum_{j=1}^n w_{ji} d(X_j, P_i)$

subject to $\sum_{j=1}^n w_{ji} = w_i, \quad i = 1, \dots, m$

$w_{ji} \geq 0, \quad j = 1, \dots, n; i = 1, \dots, m$

where $X = [X_j] = [(x_j, y_j)]$, $j = 1, \dots, n$, NF locations
 $W = [w_{ji}]$, $j = 1, \dots, n; i = 1, \dots, m$, allocated flow requirements
 $P_i = (a_i, b_i)$, location of EF i
 $d(X_j, P_i)$ = distance between NF j and EF i
 w_i = flow requirement of EF i

Since there are no capacity constraints on the NFs, optimal solutions lie at extreme points of the constraint set of the nonlinear programming (Continuous) formulation of LA problem, i.e.,

$$w_{ki} = w_i, \text{ for } j = k, \text{ and } w_{ji} = 0, \text{ for } j \neq k,$$

allocated flow requirements W can be replaced by the allocation vector

$$\alpha = [\alpha_i], \quad i = 1, \dots, m, \quad \text{and} \quad \alpha_i \in \{1, \dots, n\}$$

resulting in a mixed continuous–combinatorial formulation:

Mixed: Minimize $f(X, \alpha) = \sum_{i=1}^m w_i d(X_{\alpha_i}, P_i)$

If there were constraints on the maximum flow capacity of the NFs, then more than one w_{ji} could be nonzero in an optimal solution and W could not be replaced by the allocation vector α .

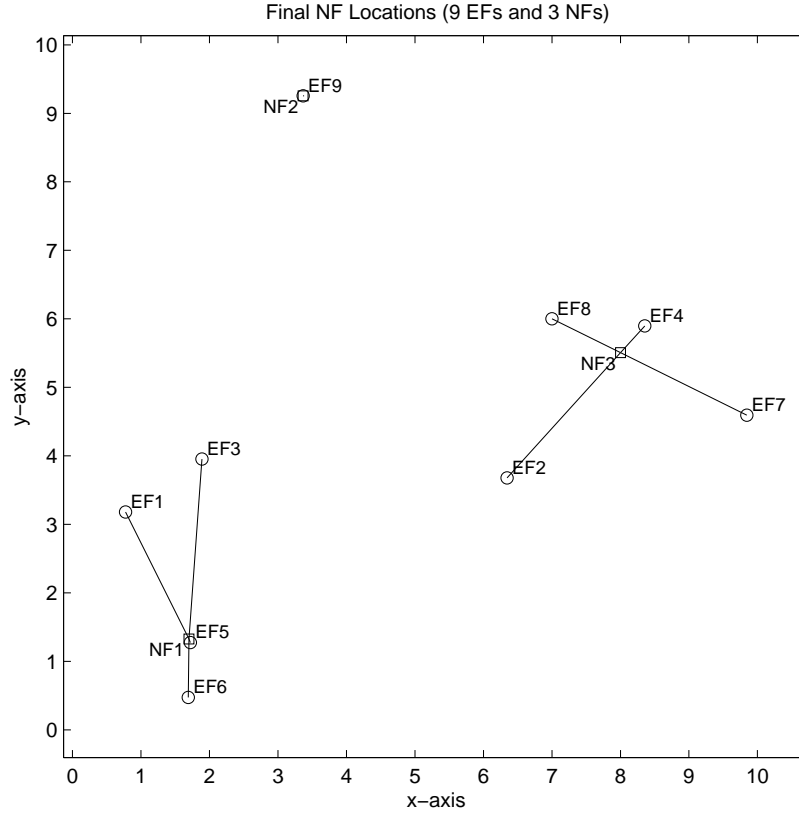


Figure 1. 9-EF by 3-NF location–allocation problem instance.

$$\text{NF locations: } X = [(1.7, 1.3), (3.4, 9.3), (8.0, 5.5)]$$

$$\text{Allocation vector: } \alpha = [1 \ 3 \ 1 \ 3 \ 1 \ 1 \ 3 \ 3 \ 2]$$

$$\text{Total transportation cost: } f(X, \alpha) = 11.78$$

$$\text{Euclidean distances: } d(X_{\alpha_i}, P_i) = \sqrt{(x_{\alpha_i} - a_i)^2 + (y_{\alpha_i} - b_i)^2}$$

$$\text{Flow requirements: All } w_i = 1$$

$$\text{Number of feasible allocations: } \begin{Bmatrix} m \\ n \end{Bmatrix} = \begin{Bmatrix} 9 \\ 3 \end{Bmatrix} = 3,025$$

Alternate Location–Allocation (ALA) Procedure

Given initial NF locations, ALA local improvement procedure finds optimal EF allocations and then finds optimal NF locations for these allocations, continuing to alternate until no further EF allocation changes are made. Introduced by Cooper in 1963, ALA is still the best heuristic for the LA problem. Since the ALA procedure finds only a local optima, the procedure should be applied multiple times using different initial NF locations, keeping the best solution found as the final solution. This type of repeated application of a local improvement procedure is termed a *multistart metaheuristic*.

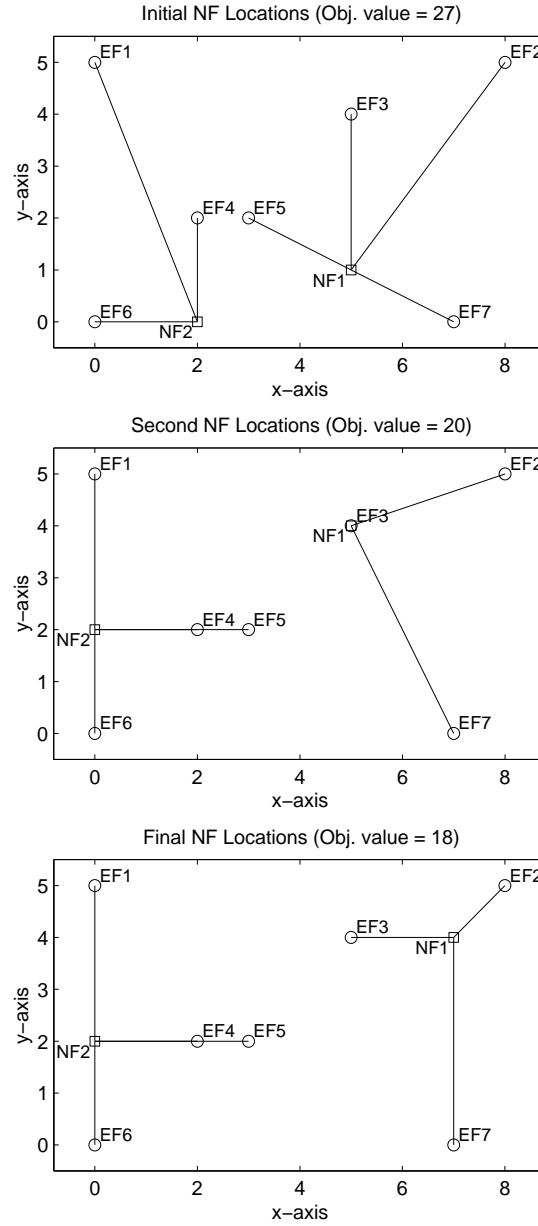


Figure 2. ALA procedure for 7-EF by 2-NF problem instance

ALA PROCEDURE: $(X, \alpha) \leftarrow ALA(X, P, w)$

1. Given initial NF locations X
2. $TC \leftarrow \infty$
3. $\alpha' \leftarrow allocate(X, P)$
4. $X' \leftarrow locate(\alpha', P, w)$
5. If $TC(X', \alpha') > TC$, stop; otherwise
 $TC \leftarrow TC(X', \alpha')$, $X \leftarrow X'$, $\alpha \leftarrow \alpha'$, and go to step 3

LOCATION PROCEDURE:

- Solve n single-facility location problems using the EFs allocated to each NF
- Exact $O(m)$ procedure for rectilinear distances (median conditions)
- Iterative procedure for general l_p distances—in MATLAB, quasi-Newton followed by Nelder-Mead simplex

ALLOCATION PROCEDURE:

- Allocate each EF to its closest NF
- If NFs were capacitated, then would have to solve a minimum cost network flow problem to perform the allocation, where each EF might be allocated to more than one NF

Solution Space of LA Problem

Mix of continuous and combinatorial

Continuous: X

$2n$ -dimensional space of NF locations: $X = [(x_1, y_1), \dots, (x_n, y_n)]$

Combinatorial: α

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \frac{1}{n!} \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} j^m \text{ feasible allocations of } n \text{ NFs to } m \text{ EFs}$$

Although n^m allocations are possible, since NFs are indistinguishable:

Number of feasible allocations = number of ways m distinguishable EFs can be allocated to n indistinguishable NFs, with each NF allocated to at least one EF

= Stirling number of the second kind

$$= \left\{ \begin{matrix} m \\ n \end{matrix} \right\}$$

For example: $\left\{ \begin{matrix} 7 \\ 2 \end{matrix} \right\} = 63$, $\left\{ \begin{matrix} 20 \\ 3 \end{matrix} \right\} = 5.8 \times 10^8$, and $\left\{ \begin{matrix} 65 \\ 5 \end{matrix} \right\} = 2.3 \times 10^{43}$