Facility Location–Allocation Problem

Location–Allocation (LA) Problem: Determine both the location of n new facilities (NFs) and the allocation of the flow requirements of m existing facilities (EFs) to the NFs that minimize total transportation costs.

Continuous: Minimize $f\left(X,W\right)=\sum_{i=1}^{m}\sum_{j=1}^{n}w_{ji}d(X_j,P_i)$

$$\text{subject to} \quad \sum_{j=1}^{n}w_{ji}=w_i, \quad i=1,\ldots,m$$

$$w_{ji}\geq 0, \quad j=1,\ldots,n;\ i=1,\ldots,m$$

where $X=\left[X_j\right]=\left[\left(x_j,y_j\right)\right], \quad j=1,\ldots,n$, NF locations

$W=\left[w_{ji}\right], \quad j=1,\ldots,n;\ i=1,\ldots,m$, allocated flow requirements

$P_i=\left(a_i,b_i\right)$, location of EF i

$d(X_j,P_i)=$ distance between NF j and EF i

$w_i=$ flow requirement of EF i

Since there are no capacity constraints on the NFs, optimal solutions lie at extreme points of the constraint set of the nonlinear programming (Continuous) formulation of LA problem, i.e.,

$$w_{ki}=w_i, \text{ for } j=k, \text{ and } w_{ji}=0, \text{ for } j\neq k,$$

allocated flow requirements W can be replaced by the allocation vector

$$\alpha=\left[\alpha_i\right], \quad i=1,\ldots,m, \quad \text{and} \quad \alpha_i\in\left\{1,\ldots,n\right\}$$

resulting in a mixed continuous–combinatorial formulation:

Mixed: Minimize $f\left(X,\alpha\right)=\sum_{i=1}^{m}w_i d(X_{\alpha_i},P_i)$

If there were constraints on the maximum flow capacity of the NFs, then more than one w_{ji} could be nonzero in an optimal solution and W could not be replaced by the allocation vector α.

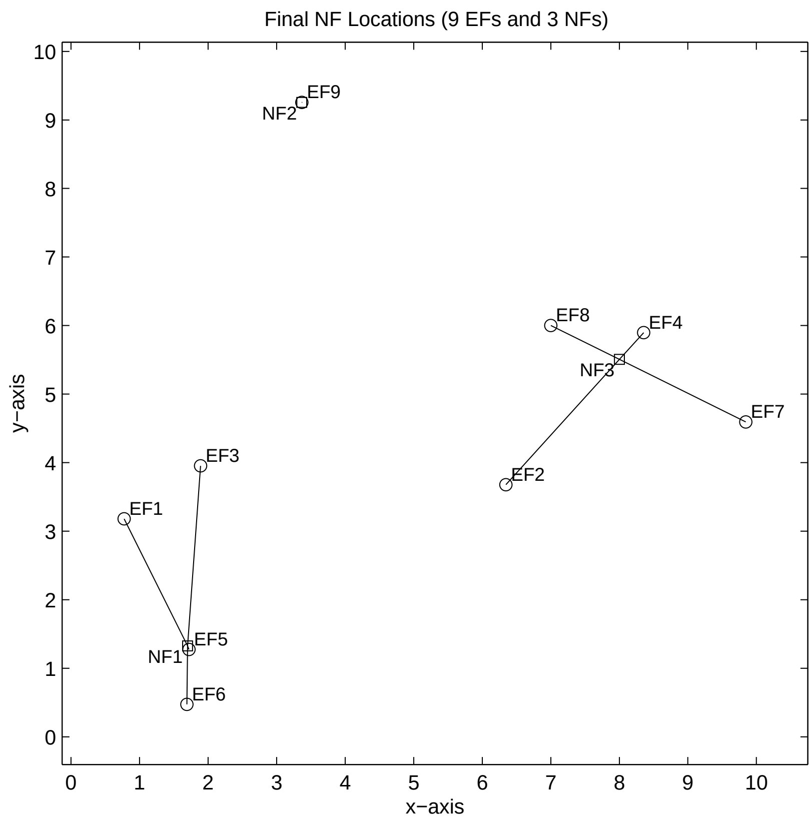

Figure 1. 9-EF by 3-NF location–allocation problem instance.

NF locations: $X = \left[(1.7, 1.3), (3.4, 9.3), (8.0, 5.5) \right]$

Allocation vector: $\alpha = \left[1 \quad 3 \quad 1 \quad 3 \quad 1 \quad 1 \quad 3 \quad 3 \quad 2 \right]$

Total transportation cost: $f\left(X, \alpha\right) = 11.78$

Euclidean distances: $d\left(X_{\alpha_i}, P_i\right) = \sqrt{\left(x_{\alpha_i} - a_i\right)^2 + \left(y_{\alpha_i} - b_i\right)^2}$

Flow requirements: All $w_i = 1$

Number of feasible allocations: $\begin{Bmatrix} m \\ n \end{Bmatrix} = \begin{Bmatrix} 9 \\ 3 \end{Bmatrix} = 3,025$

Alternate Location–Allocation (ALA) Procedure

Given initial NF locations, ALA local improvement procedure finds optimal EF allocations and then finds optimal NF locations for these allocations, continuing to alternate until no further EF allocation changes are made. Introduced by Cooper in 1963, ALA is still the best heuristic for the LA problem. Since the ALA procedure finds only a local optima, the procedure should be applied multiple times using different initial NF locations, keeping the best solution found as the final solution. This type of repeated application of a local improvement procedure is termed a *multistart metaheuristic*.

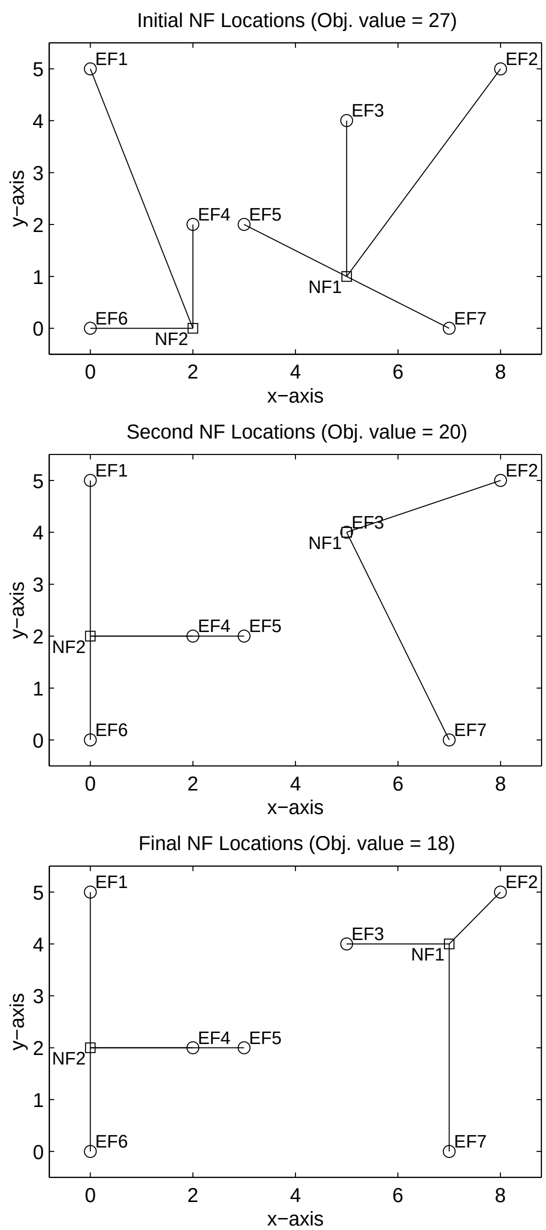

Figure 2. ALA procedure for 7-EF by 2-NF problem instance

ALA PROCEDURE: $(X,\alpha) \leftarrow ALA(X,P,w)$

1. Given initial NF locations X
2. $TC \leftarrow \infty$
3. $\alpha' \leftarrow allocate(X,P)$
4. $X' \leftarrow locate(\alpha',P,w)$
5. If $TC(X',\alpha') > TC$, stop; otherwise
   $TC \leftarrow TC(X',\alpha')$, $X \leftarrow X'$, $\alpha \leftarrow \alpha'$, and go to step 3

LOCATION PROCEDURE:

- Solve n single-facility location problems using the EFs allocated to each NF
- Exact $O(m)$ procedure for rectilinear distances (median conditions)
- Iterative procedure for general l_p distances—in MATLAB, quasi-Newton followed by Nelder-Mead simplex

ALLOCATION PROCEDURE:

- Allocate each EF to its closest NF
- If NFs were capacitated, then would have to solve a minimum cost network flow problem to perform the allocation, where each EF might be allocated to more than one NF

Solution Space of LA Problem

Mix of continuous and combinatorial

Continuous: X

$$2n\text{-dimensional space of NF locations: } X = \left[(x_1, y_1), \ldots, (x_n, y_n) \right]$$

Combinatorial: α

$$\left\{ \begin{matrix} m \\ n \end{matrix} \right\} = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} j^m \text{ feasible allocations of } n \text{ NFs to } m \text{ EFs}$$

Although n^m allocations are possible, since NFs are indistinguishable:

$$\begin{aligned} \text{Number of feasible allocations} &= \text{number of ways } m \text{ distinguishable EFs can be allocated to } n \text{ indistinguishable NFs, with each NF allocated to at least one EF} \\ &= \text{Stirling number of the second kind} \\ &= \left\{ \begin{matrix} m \\ n \end{matrix} \right\} \end{aligned}$$

For example: $\left\{ \begin{matrix} 7 \\ 2 \end{matrix} \right\} = 63$, $\left\{ \begin{matrix} 20 \\ 3 \end{matrix} \right\} = 5.8 \times 10^8$, and $\left\{ \begin{matrix} 65 \\ 5 \end{matrix} \right\} = 2.3 \times 10^{43}$