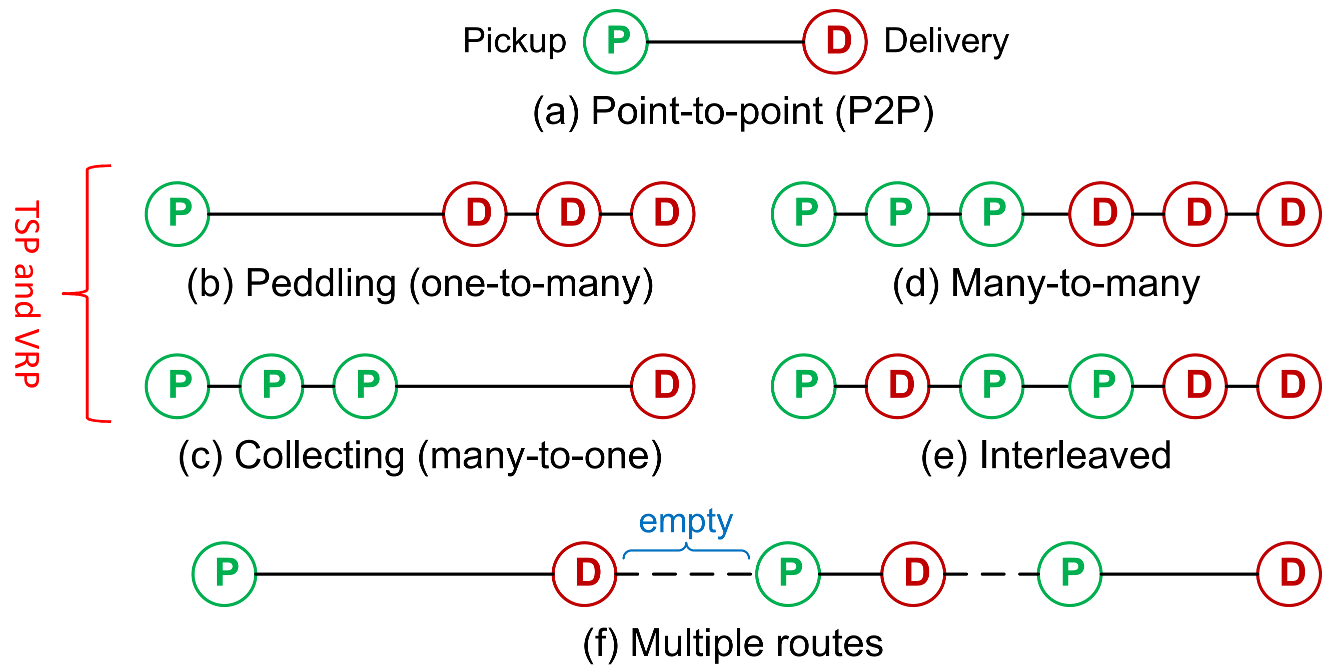
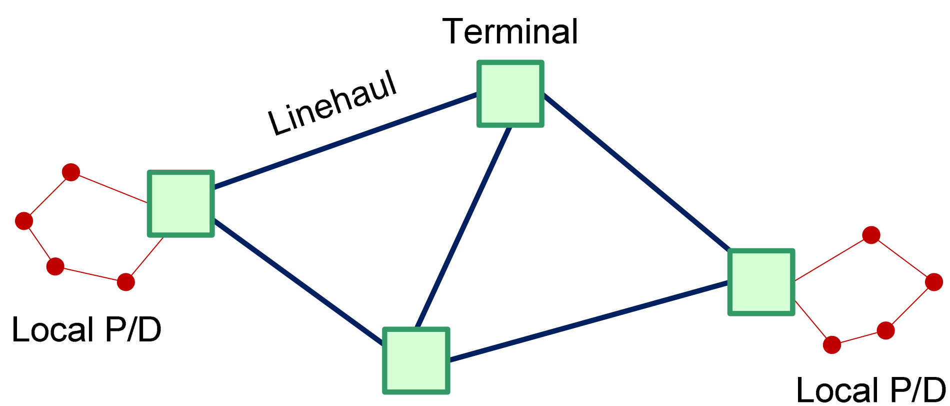
Routing 1: Traveling Salesman Problem

- Well solved problem
  - in practice and in approximation
  - not in theory

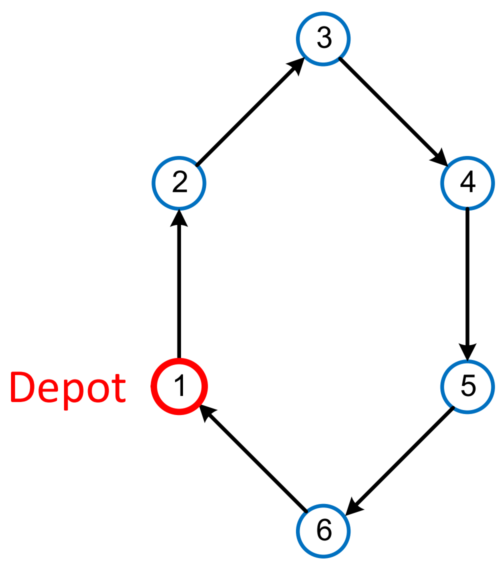
Routing Alternatives





TSP

- Problem: find connected sequence through all nodes of a graph that minimizes total arc cost
  - Termed a *tour* since returns to starting node
  - Subroutine in most vehicle routing problems
  - Node sequence can represent a route only if all pickups and/or deliveries occur at a single node (depot)

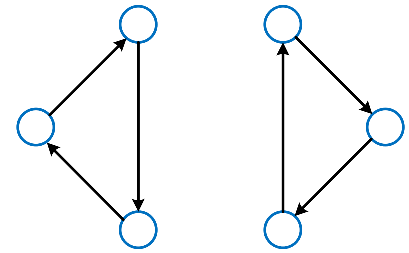


Node sequence = permutation + start node

1	2	3	4	5	6	1

$n = 6 \Rightarrow (n-1)! = 120 \quad \text{possible solutions}$

TSP



- TSP can be solved by a mix of *construction* and *improvement* procedures
  - BIP formulation has an exponential number of constraints to eliminate subtours ($\Rightarrow$ column generation techniques)
- Asymmetric: only best-known solutions for large *n*

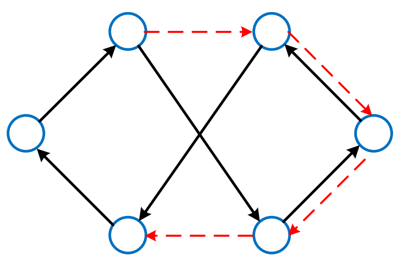
$$(n-1)! \quad n = 13 \Rightarrow \approx \frac{1}{2} \text{billion solutions}$$

Note: $n-1$ because starting node of a route is arbitrary

- Symmetric: solved to optimal using BIP

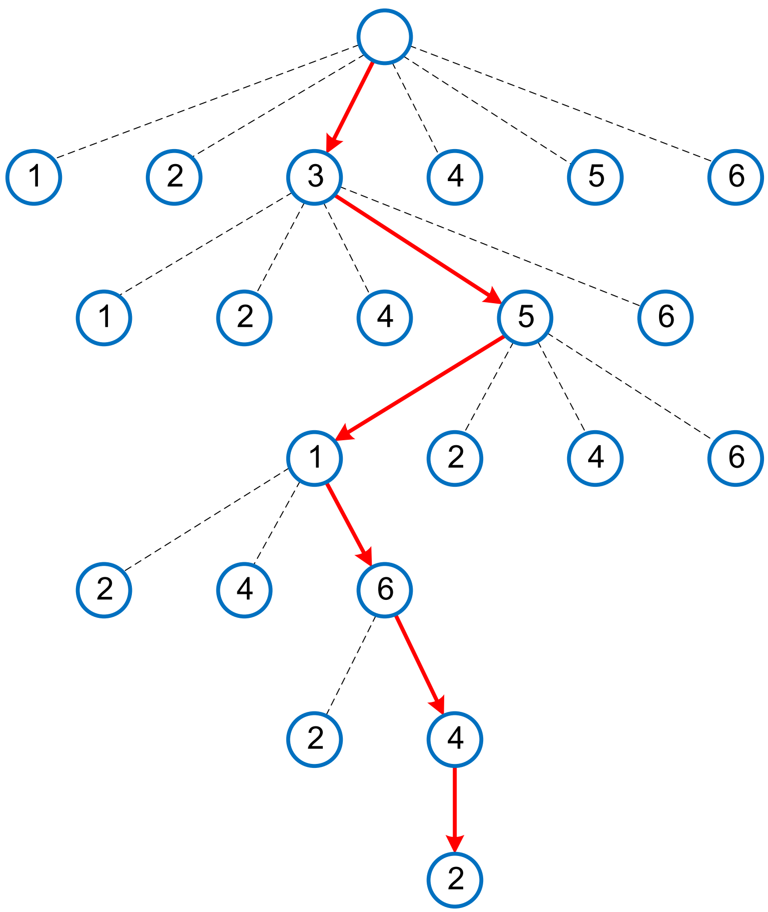
$$c_{ij} = c_{ji} \Rightarrow \frac{(n-1)!}{2} \quad \text{solutions}$$

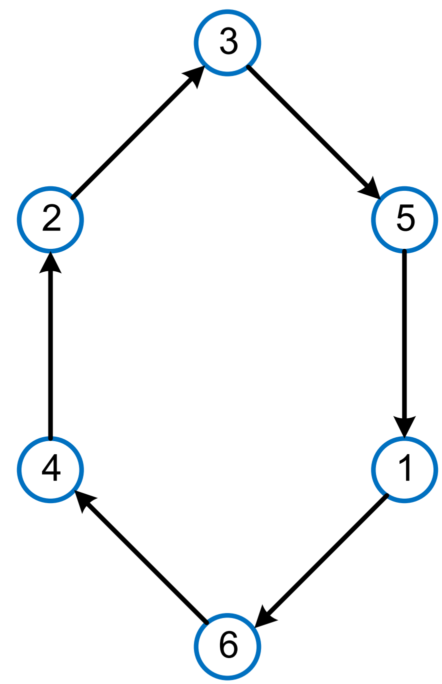
- Euclidean: arcs costs = distance between nodes



TSP Construction

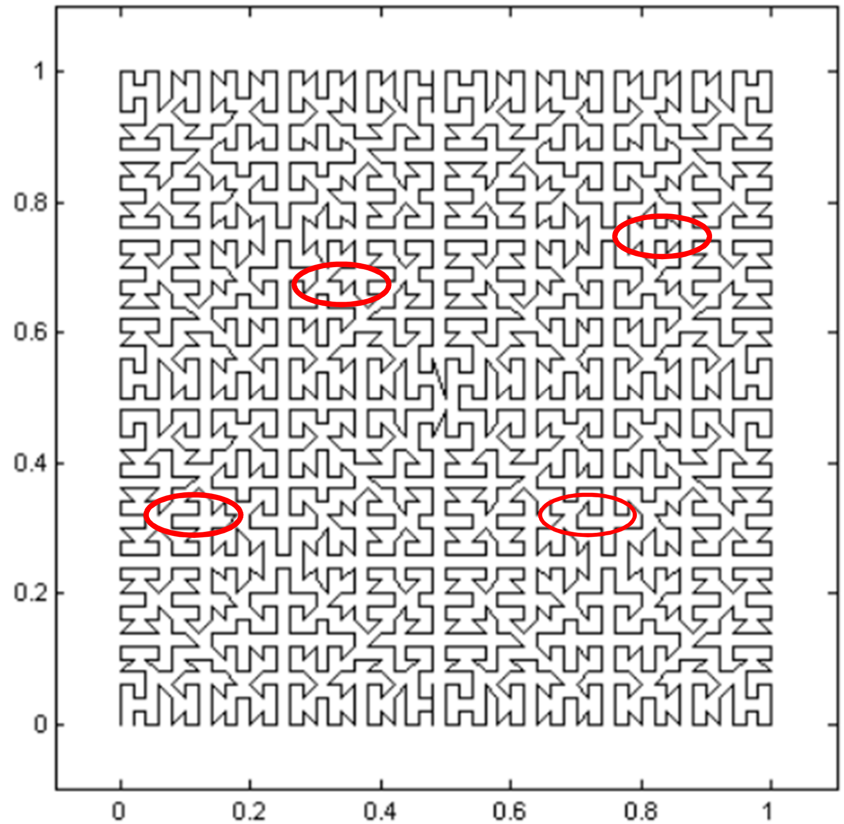
- Construction easy since any permutation is feasible and can then be improved





Spacefilling Curve

1.0	0.250	0.254	0.265	0.298	0.309	0.438	0.441	0.452	0.485	0.496	0.500
0.9	0.246	0.257	0.271	0.292	0.305	0.434	0.445	0.458	0.479	0.493	0.504
0.8	0.235	0.229	0.279	0.283	0.333	0.423	0.417	0.467	0.471	0.521	0.515
0.7	0.202	0.208	0.158	0.154	0.354	0.390	0.396	0.596	0.592	0.542	0.548
0.6	0.191	0.180	0.167	0.146	0.132	0.379	0.618	0.604	0.583	0.570	0.559
0.5	0.188	0.184	0.173	0.140	0.129	0.375	0.621	0.610	0.577	0.566	0.563
0.4	0.059	0.070	0.083	0.104	0.118	0.871	0.632	0.646	0.667	0.680	0.691
0.3	0.048	0.042	0.092	0.096	0.896	0.860	0.854	0.654	0.658	0.708	0.702
0.2	0.015	0.021	0.971	0.967	0.917	0.827	0.833	0.783	0.779	0.729	0.735
0.1	0.004	0.993	0.979	0.958	0.945	0.816	0.805	0.792	0.771	0.757	0.746
0.0	0.000	0.996	0.985	0.952	0.941	0.813	0.809	0.798	0.765	0.754	0.750
θ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

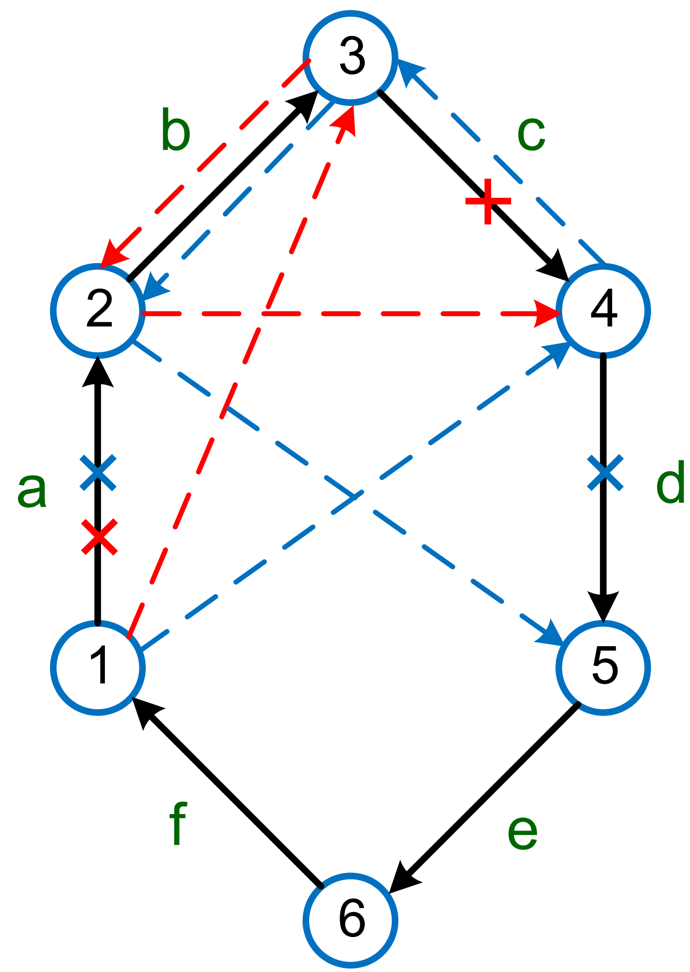


Sequence determined by sorting position along 1-D line covering 2-D space

2: 0.021
3: 0.154
1: 0.471
4: 0.783

Bartholdi, John J., and Loren K. Platzman. "Heuristics Based on Spacefilling Curves for Combinatorial Problems in Euclidean Space." *Management Science*, vol. 34, no. 3, 1988, pp. 291–305. *JSTOR*, www.jstor.org/stable/2632046. Accessed 20 Oct. 2020.

Two-Opt Improvement



	a	b	c	d	e	f	
	1	2	3	4	5	6	1
a-c	1	3	2	4	5	6	1
a-d	1	4	3	2	5	6	1
a-e	1	5	4	3	2	6	1
b-d							
b-e							
b-f							
c-e							
c-f							
d-f							

Sequences considered at end to verify local optimum: n nodes $\Rightarrow \overbrace{\sum_{j=3}^{n-1}(1)}^{\text{first arc } a} + \overbrace{\sum_{i=2}^{n-2}\sum_{j=i+2}^{n}}^{\text{arcs } b \text{ to } n-2}(1) = \frac{n(n-3)}{2} = 9 \text{ for } n = 6$

Ex: Two-Opt Improvement

- Order in which *twoopt* considers each sequence:

1:	1	2	3	4	5	6	1	38
2:	1	3	2	4	5	6	1	39
3:	1	4	3	2	5	6	1	32
4:	1	3	4	2	5	6	1	31
5:	1	4	3	2	5	6	1	32
6:	1	2	4	3	5	6	1	31
7:	1	5	2	4	3	6	1	21
8:	1	2	5	4	3	6	1	21
9:	1	4	2	5	3	6	1	32
10:	1	3	4	2	5	6	1	31
11:	1	5	4	2	3	6	1	12
12:	1	4	5	2	3	6	1	34
13:	1	2	4	5	3	6	1	40
14:	1	3	2	4	5	6	1	39
15:	1	5	2	4	3	6	1	21
16:	1	5	3	2	4	6	1	30
17:	1	5	6	3	2	4	1	31
18:	1	5	4	3	2	6	1	13
19:	1	5	4	6	3	2	1	18
20:	1	5	4	2	6	3	1	20

Local optimal sequence

Sequences considered at end to verify local optimum: n nodes $\Rightarrow$

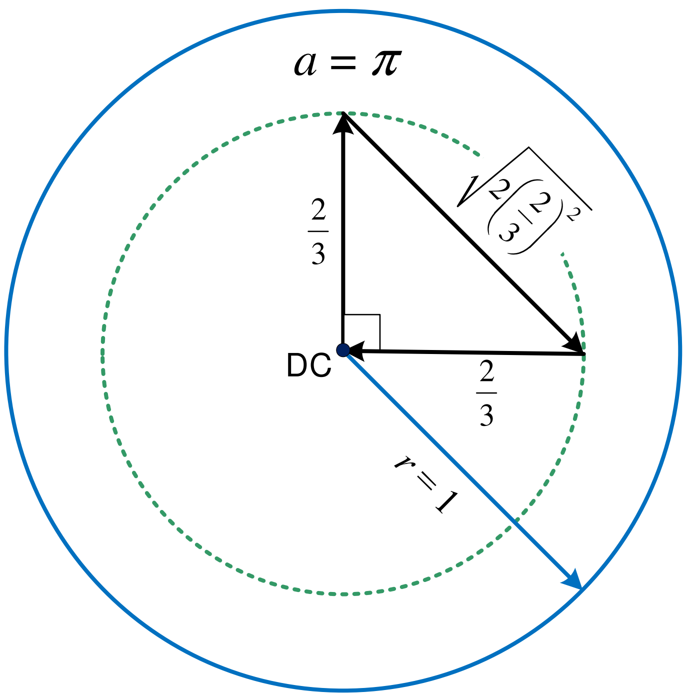
$$\sum_{j=3}^{n-1}(1)+\sum_{i=2}^{n-2}\sum_{j=i+2}^{n}(1)=\frac{n(n-3)}{2}=9 \text{ for } n=6$$

D:	1	2	3	4	5	6
1:	0	8	6	9	1	5
2:	3	0	1	5	4	2
3:	9	2	0	3	1	1
4:	8	2	1	0	10	6
5:	6	7	10	1	0	10
6:	6	2	5	2	1	0

Note: Not symmetric

Expected Two-Customer Route Distance

Know: the expected distance from center of a circle, $d_a \approx \dfrac{2r}{3}$, and the expected angle between angle between two points is 90° (?); let area $a = \pi$ so that radius $r = 1$ and expected distance of an $m = 2$ customer route is $2\left(\dfrac{2}{3}\right)+\sqrt{2\left(\dfrac{2}{3}\right)^2} = \dfrac{2}{3}\left(2+\sqrt{2}\right)$.




```
span = 360; n = 1e6;
x = rand(n,1)*span; y = rand(n,1)*span;
absDiffDeg = @(a,b) min(360-mod(a-b,360), mod(a-b,360));
ang = absDiffDeg(x,y); % Abs difference between two angles
vdisp('min(ang),max(ang),mean(ang)')

 :  min(ang)  max(ang)  mean(ang)
-:------------------------------
1:   0.0000    180.00     90.02
```

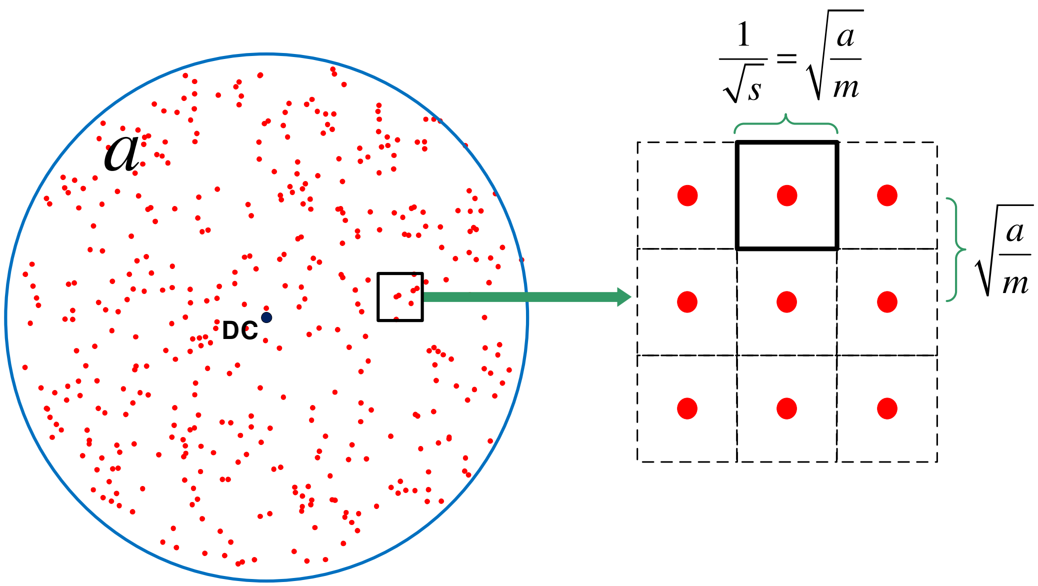
```
span = 180; n = 1e6;
x = rand(n,1)*span; y = rand(n,1)*span;
absDiffDeg = @(a,b) min(360-mod(a-b,360), mod(a-b,360));
ang = absDiffDeg(x,y); % Abs difference between two angles
vdisp('min(ang),max(ang),mean(ang)')

 :  min(ang)  max(ang)  mean(ang)
-:------------------------------
1:   0.0002    179.64     59.98
```

Expected Route Distance

Given m customers in an area a, the density is $s = m/a$, and the expected distance between customers is $s^{-0.5} = \sqrt{a/m}$, resulting in an estimated total route distance that is proportional to $\varphi m\sqrt{a/m}$. Use known route distance for $m = 2$ to determine φ:

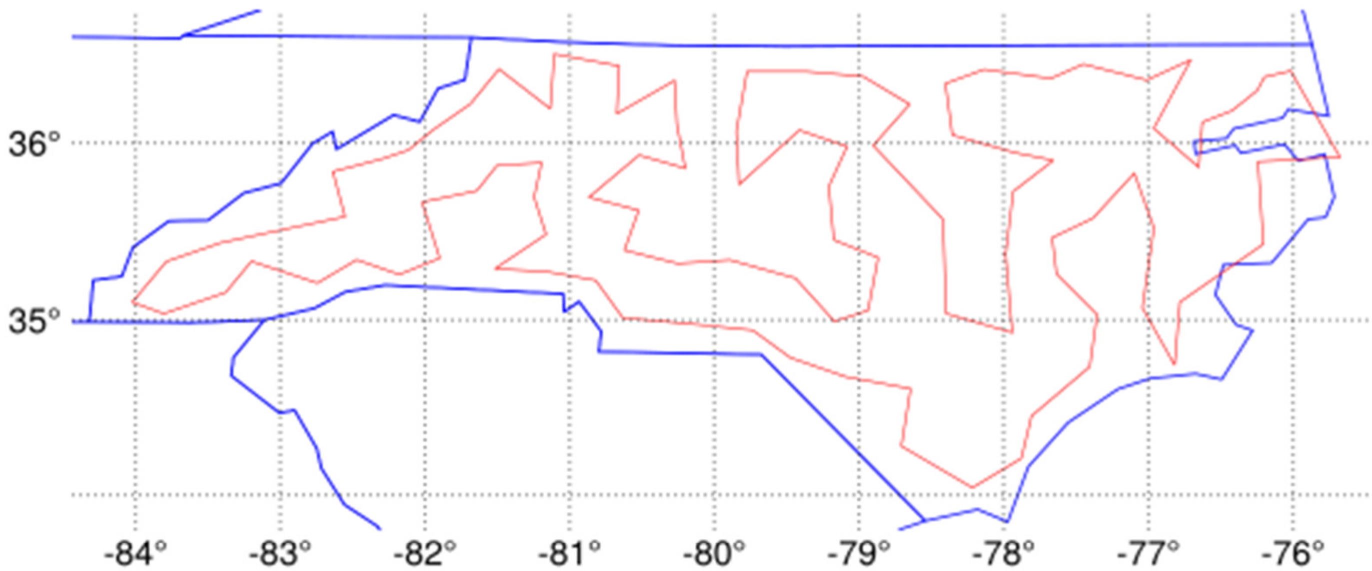
For $a = \pi$ and $m = 2$, $\varphi 2\sqrt{\dfrac{\pi}{2}} = \dfrac{2}{3}\left(2+\sqrt{2}\right) \Rightarrow \varphi = \dfrac{2\left(\sqrt{2}+1\right)}{3\sqrt{\pi}} \approx 0.9$, so that

$\boxed{\hat{d}_m^{TSP} = 0.9\sqrt{ma}}$, for routes passing through the center (DC) of a circular region.



:	m	Simulated	Estimate
1:	2	2.26	2.26
2:	5	3.79	3.57
3:	10	5.20	5.04
4:	20	7.02	7.13
5:	50	10.95	11.28
6:	100	15.50	15.95
7:	200	22.09	22.56
8:	500	35.07	35.67
9:	1,000	49.87	50.44

Ex: Tour of NC



Expected Tour Distance

```
@show a = sum(df.ALAND) + sum(df.AWATER)
@show TCᴱ = 0.9sqrt(nrow(df)*a)
@show TCᵒ
TCᴱ/TCᵒ
```

```
a = sum(df.ALAND) + sum(df.AWATER) = 53818.558999999994
TCᴱ = 0.9 * sqrt(nrow(df) * a) = 2087.8944606947925
TCᵒ = 2141.2129756166273
```

```
0.9750989203180594
```